NATIONAL BOARD FOR HIGHER MATHEMATICS

M. A. and M.Sc. Scholarship Test

September 22, 2007

Time Allowed: 150 Minutes

Maximum Marks: 45

Please read, carefully, the instructions on the following page

INSTRUCTIONS TO CANDIDATES

- Please ensure that this question paper booklet contains 11 numbered (and printed) pages. The reverse of each printed page is blank and can be used for rough work.

- There are three parts to this test: Algebra, Analysis and Geometry. Each part consists of **15** questions adding up to **45** questions in all.

- Answer each question, as directed, in the space provided for it in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so do not answer questions on it.

- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a),(b) and (c)).

- Points will be awarded in the above questions only if **all** the correct choices are made. There will be no partial credit.

- $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively. The symbol I will denote the identity matrix of appropriate order.

Section 1: Algebra

1.1 Let A be the matrix

$$A = \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -1 \end{pmatrix}$$

Compute the matrix $B = 3A - 2A^2 - A^3 - 5A^4 + A^6$.

1.2 How many elements of order 2 are there in the group

$$(\mathbb{Z}/4\mathbb{Z})^3?$$

1.3 Consider the permutation π given by

$$\begin{array}{llllllllllll} n & = 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \pi(n) & = 5 & 7 & 8 & 10 & 6 & 1 & 2 & 4 & 9 & 3 \end{array}$$

Find the order of the permutation π.

1.4 Consider the system of simultaneous equations

$$\begin{array}{rrrcl} 2x & -2y & -2z & = & a_1 \\ -2x & +2y & -3z & = & a_2 \\ 4x & -4y & +5z & = & a_3 \end{array}$$

Write down the condition to be satisfied by a_1, a_2, a_3 for this system NOT to have a solution.

1.5 Write down a polynomial of degree 4 with integer coefficients which has $\sqrt{3} + \sqrt{5}$ as a root.

1.6 A finite group G acts on a finite set X, the action of $g \in G$ on $x \in X$ being denoted by gx. For each $x \in X$ the stabiliser at x is the subgroup $G_x = \{g \in G : gx = x\}$. If $x, y \in G$ and if $y = gx$, then express G_y in terms of G_x.

1.7. Write down the last two digits of 9^{1500}.

1.8 A permutation matrix A is a *nonsingular* square matrix in which each row has exactly one entry $= 1$, the other entries being all zeros. If A is an $n \times n$ permutation matrix, what are the possible values of determinant of A?

1.9 Let V be the vector space of all polynomials of degree at most equal to $2n$ with real coefficients. Let V_0 stand for the vector subspace $V_0 = \{P \in V : P(1) + P(-1) = 0\}$ and V_e stand for the subspace of polynomials which have terms of even degree alone. If $\dim(U)$ stands for the dimension of a vector space U, then find $\dim(V_0)$ and $\dim(V_0 \cap V_e)$.

1.10 Let a, b, m and n be integers, m, n positive, $am + bn = 1$. Find an integer x (in terms of a, b, m, n, p, q) so that

$$\begin{aligned} x &\equiv p \pmod{m} \\ x &\equiv q \pmod{n} \end{aligned}$$

where p and q are given integers.

1.11 In the ring $\mathbb{Z}/20\mathbb{Z}$ of integers modulo 20, does the equivalence class $\overline{17}$ have a multiplicative inverse? Write down an inverse if your answer is yes.

1.12 Let $\mathbb{R}[x]$ be the ring of polynomials in the indeterminate x over the field of real numbers and let $\mathcal{J}$ be the ideal generated by the polynomial $x^3 - x$. Find the dimension of the vector space $\mathbb{R}[x]/\mathcal{J}$.

1.13 In the ring of polynomials $R = Z_5[x]$ with coefficients from the field $\mathbb{Z}_5$, consider the smallest ideal $\mathcal{J}$ containing the polynomials,

$$\begin{aligned} p_1(x) &= x^3 + 4x^2 + 4x + 1 \\ p_2(x) &= x^2 + x + 3. \end{aligned}$$

Which of the following polynomials $q(x)$ has the property that $\mathcal{J} = q(x)R$?
(a) $q(x) = p_2(x)$
(b) $q(x) = x - 1$
(c) $q(x) = x + 1$

1.14 In how many ways can 20 indistinguishable pencils be distributed among four children A,B,C and D ?

1.15 Let $w = u+iv$ and, $z = x+iy$ be complex numbers such that $w^2 = z^2+1$. Then which of the following inequalities must always be true?
(a) $x \leq u$
(b) $y^2 \leq v^2$
(c) $v^2 \leq y^2$

Section 2: Analysis

2.1 Evaluate:

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}.$$

2.2 Evaluate:

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\sqrt{n^2-k^2}.$$

2.3 Pick out the uniformly continuous functions from the following and, in such cases, given $\varepsilon > 0$, find $\delta > 0$ explicitly as a function of ε so that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$.
(a) $f(x) = \sqrt{x},\ 1 \le x \le 2.$
(b) $f(x) = x^3,\ x \in \mathbb{R}.$
(c) $f(x) = \sin^2 x,\ x \in \mathbb{R}.$

2.4 Which of the following functions are differentiable at $x = 0$?
(a)

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

(b) $f(x) = |x|x.$
(c)

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

2.5 Find the coefficient of x^7 in the Maclaurin series expansion of the function $f(x) = \sin^{-1} x$.

2.6 Compute

$$f(x) = \lim_{n\to\infty} n^2 x(1-x^2)^n$$

where $0 \le x \le 1$.

2.7 Which of the following series are convergent?
(a)

$$\sum_{n=1}^{\infty} \sqrt{\frac{2n^2+3}{5n^3+7}}.$$

(b)

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+\frac{3}{2}}}.$$

(c)

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right).$$

2.8 Find the interval of convergence of the series:

$$\sum_{n=1}^{\infty} \frac{\log(n+1)}{\sqrt{n+1}}(x-5)^n.$$

2.9 Evaluate:

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x \, dx}{\sin x + \cos x}.$$

2.10 Examine for maxima and minima:

$$f(x,y) = x^2 + 5y^2 - 6x + 10y + 6.$$

2.11 Find the point(s) on the parabola $2x^2 + 2y = 3$ nearest to the origin.What is the shortest distance?

2.12 Let S be the triangular region in the plane with vertices at $(0,0), (1,0)$ and $(1,1)$. Let $f(x,y)$ be a continuous function. Express the double integral $\int\int_S f(x,y)\, dA$ in two different ways as iterated integrals (*i.e.* in the forms $\int_\alpha^\beta \int_{\gamma(x)}^{\delta(x)} f(x,y)\, dy\, dx$ and $\int_a^b \int_{c(y)}^{d(y)} f(x,y)\, dx\, dy$.)

2.13 Let $\omega \neq 1$ be a seventh root of unity. Write down a polynomial equation of degree ≤ 6 satisfied by ω.

2.14 Let $z = x + iy$. Which of the following functions are analytic in the entire complex plane?
(a) $f(x, y) = e^x(\cos y - i \sin y)$.
(b) $f(x, y) = e^{-x}(\cos y - i \sin y)$.
(c) $f(x, y) = \min\{2, x^2 + y^2\}$.

2.15 Let C denote the boundary of the square whose sides are given by the lines $x = \pm 2$ and $y = \pm 2$. Assume that C is described in the positive sense, *i.e.*, anticlockwise. Evaluate:

$$\int_C \frac{\cos z \; dz}{z(z^2 + 8)}.$$

Section 3: Geometry

3.1 Let A be the point $(0, 4)$ in the xy-plane and let B be the point $(2t, 0)$. Let L be the mid point of AB and let the perpendicular bisector of AB meet the y-axis at M. Let N be the mid-point of LM. Find the locus of N (as t varies).

3.2 Let $(a_1, a_2), (b_1, b_2)$ and (c_1, c_2) be three *non-collinear* points in the xy-plane. Let r, s and t be three real numbers such that (i)$r + s + t = 0$, (ii) $ra_1 + sb_1 + tc_1 = 0$ and (iii) $ra_2 + sb_2 + tc_2 = 0$. Write down all the possible values of r, s and t.

3.3 Consider the equation $2x + 4y - x^2 - y^2 = 5$. Which of the following does it represent?
(a) a circle.
(b) an ellipse.
(c) a pair of straight lines.

3.4 Write down the equations of the circles of radius 5 passing through the origin and having the line $y = 2x$ as a tangent.

3.5 Two equal sides of an isoceles triangle are given by the equations $y = 7x$ and $y = -x$. If the third side passes through the point $(1, -10)$, pick out the equation(s) which *cannot* represent that side.
(a) $3x + y + 7 = 0$.
(b) $x - 3y - 31 = 0$.
(c) $x + 3y + 29 = 0$.

3.6 Let $m \neq 0$. Consider the line $y = mx + \frac{a}{m}$ and the parabola $y^2 = 4ax$. Pick out the true statements.
(a) The line intersects the parabola at exactly one point.
(b) The line intersects the parabola at two points whenever $|m| < 2\sqrt{a}$.
(c) The line is tangent to the parabola only when $|m| = 2\sqrt{a}$.

3.7 Consider the circle $x^2 + (y + 1)^2 = 1$. Let a line through the origin O meet the circle again at a point A. Let B be a point on OA such that $OB/OA = p$, where p is a given positive number. Find the locus of B.

3.8 Let $a > 0$ and $b > 0$. Let a straight line make an intercept a on the x-axis and b on the line through the origin which is inclined at an angle θ to the x-axis, both in the first quadrant. Write down the equation of the straight line.

3.9 What does the following equation represent?

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 \; = \; 0.$$

3.10 Find the coordinates of the centre of the circumcircle of the triangle whose vertices are the points $(4, 1), (-1, 6)$ and $(-4, -3)$.

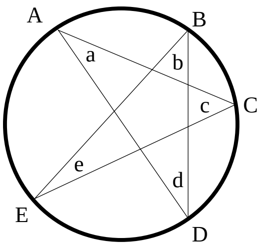
3.11 Let A and B be the points of intersection of the circles $x^2+y^2-4x-5 = 0$ and $x^2 + y^2 + 8y + 7 = 0$. Find the centre and radius of the circle whose diameter is AB.

3.12 Ten points are placed at random in the unit square. Let ρ be the minimum distance between all pairs of distinct points from this set. Find the *least upper bound* for ρ.

3.13 Let K be a subset of the plane. It is said to be *convex* if given any two points in K, the line segment joining them is also contained in K. It is said to be *strictly convex* if given any two points in K, the mid-point of the line segment joining them lies in the *interior* of K. In each of the following cases determine whether the given set is convex (but not strictly convex), strictly convex or not convex.
(a) $K = \{(x, y) \mid x^2 + y^2 \leq 1\}$.
(b) $K = \{(x, y) \mid |x| + |y| \leq 1\}$.
(c) $K = \{(x, y) \mid x^{\frac{2}{3}} + y^{\frac{2}{3}} \leq 1\}$.

3.14 Consider the set $K = \{(x, y) \mid |x| + |y| \leq 1\}$ in the plane. Given a point A in the plane, let $P_K(A)$ be the point in K which is closest to A. Let $B = (1, 0) \in K$. Determine the set

$$S \; = \; \{A \mid P_K(A) = B\}.$$



3.15 Let A, B, C, D and E be five points on a circle and let a, b, c, d and e be the angles as shown in the figure above. Which of the following equals the ratio AD/BE?

(a) $\frac{\sin(a+d)}{\sin(b+e)}$.

(b) $\frac{\sin(b+c)}{\sin(c+d)}$.

(c) $\frac{\sin(a+b)}{\sin(b+c)}$.