NATIONAL BOARD FOR HIGHER MATHEMATICS

M. A. and M.Sc. Scholarship Test

September 20, 2008

Time Allowed: 150 Minutes

Maximum Marks: 30

Please read, carefully, the instructions on the following page

INSTRUCTIONS TO CANDIDATES

- Please ensure that this question paper booklet contains 8 numbered (and printed) pages. The reverse of each printed page is blank and can be used for rough work.
- There are three parts to this test: Algebra, Analysis and Geometry. Each part consists of **10** questions adding up to **30** questions in all.
- Answer each question, as directed, in the space provided for it in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so do not answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a),(b) and (c)).
- Points will be awarded in the above questions only if **all** the correct choices are made. There will be no partial credit.
- $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively. The symbol I will denote the identity matrix of appropriate order. All logarithms, unless specified otherwise, are to the base e.
- **Calculators are not allowed.**

Section 1: Algebra

1.1 Let α, β and γ be the roots of the polynomial

$$x^3 + 2x^2 - 3x - 1.$$

Compute:

$$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}.$$

1.2 Let G be a cyclic group of order 8. How many of the elements of G are generators of this group?

1.3 Which of the following statements are true?
(a) Any group of order 15 is abelian.
(b) Any group of order 25 is abelian.
(c) Any group of order 55 is abelian.

1.4 A real number is said to be algebraic if it is the root of a non-zero polynomial with integer coefficients. Which of the following real numbers are algebraic?
(a) $\cos\frac{2\pi}{5}$
(b) $e^{\frac{1}{2}\log 2}$
(c) $5^{\frac{1}{7}} + 7^{\frac{1}{5}}$

1.5 Let $\mathbb{Z} + \sqrt{3}\mathbb{Z}$ denote the ring of numbers of the form $a + b\sqrt{3}$, where a and $b \in \mathbb{Z}$. Find the condition that $a + b\sqrt{3}$ is a unit in this ring.

1.6 Let $\mathbb{F}_p$ denote the field $\mathbb{Z}/p\mathbb{Z}$, where p is a prime. Let $\mathbb{F}_p[x]$ be the associated polynomial ring. Which of the following quotient rings are fields?
(a) $\mathbb{F}_5[x]/\{x^2 + x + 1\}$
(b) $\mathbb{F}_2[x]/\{x^3 + x + 1\}$
(c) $\mathbb{F}_3[x]/\{x^3 + x + 1\}$

1.7 Let G denote the group of invertible 2×2 matrices with entries from $\mathbb{F}_2$ (the group operation being matrix multiplication). What is the order of G?

1.8 Let A be a 3×3 upper triangular matrix with real entries. If $a_{11} = 1, a_{22} = 2$ and $a_{33} = 3$, determine α, β and γ such that

$$A^{-1} = \alpha A^2 + \beta A + \gamma I.$$

1.9 Let V be a vector space such that $\dim(V) = 5$. Let W and Z be subspaces of V such that $\dim(W) = 3$ and $\dim(Z) = 4$. Write down all possible values of $\dim(W \cap Z)$.

1.10 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which maps each point in $\mathbb{R}^2$ to its reflection on the x-axis. What is the determinant of T? What is its trace?

Section 2: Analysis

2.1 Evaluate:

$$\lim_{x\to 0}(1-\sin x\cos x)^{\mathrm{cosec} 2x}.$$

2.2 Evaluate:

$$\lim_{n\to\infty}\frac{1}{n^6}\sum_{k=1}^{n}k^5.$$

2.3 Determine if each of the following series is absolutely convergent, conditionally convergent or divergent:
(a)

$$\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2},\ \ x\in\mathbb{R}.$$

(b)

$$\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+2}.$$

(c)

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n+3}.$$

2.4 Let $f:\mathbb{R}^2\to\mathbb{R}$ be a mapping such that $f(0,0)=0$. Determine which of the following are jointly continuous at $(0,0)$:
(a)

$$f(x,y)\ =\ \frac{x^2y^2}{x^2+y^2},\ \ (x,y)\neq(0,0).$$

(b)

$$f(x,y)\ =\ \frac{xy}{x^2+y^2},\ \ (x,y)\neq(0,0).$$

(c)

$$f(x,y)\ =\ \begin{cases} x\sin\frac{1}{y}+y\sin\frac{1}{x} & \text{if } xy\neq 0\\ 0 & \text{otherwise.}\end{cases}$$

2.5 Which of the following functions are uniformly continuous?
(a) $f(x)=\sin^2 x,\ \ x\in\mathbb{R}$.
(b) $f(x)=x\sin\frac{1}{x},\ \ x\in]0,1[$.
(c) $f(x)=x^2,\ \ x\in\mathbb{R}$.

2.6 Which of the following maps are differentiable everywhere?
(a) $f(x) = |x|^3 x,\ x \in \mathbb{R}$.
(b) $f : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| \le |x - y|^{\sqrt{2}}$ for all x and $y \in \mathbb{R}$.
(c) $f(x) = x^3 \sin \frac{1}{\sqrt{|x|}}$ when $x \neq 0$ and $f(0) = 0$.

2.7 Pick out the true statements:
(a) If the series $\sum_n a_n$ and $\sum_n b_n$ are convergent, then $\sum_n a_n b_n$ is also convergent.
(b) If the series $\sum_n a_n$ is convergent and if $\sum_n b_n$ is absolutely convergent, then $\sum_n a_n b_n$ is absolutely convergent.
(c) If the series $\sum_n a_n$ is convergent, $a_n \geq 0$ for all n, and if the sequence $\{b_n\}$ is bounded, then $\sum_n a_n b_n$ is absolutely convergent.

2.8 Write down an equation of degree four satisfied by all the complex fifth roots of unity.

2.9 Evaluate:

$$2 \sin\left(\frac{\pi}{2} + i\right)$$

2.10 Let Γ be a simple closed contour in the complex plane described in the positive sense. Evaluate

$$\int_\Gamma \frac{z^3 + 2z}{(z - z_0)^3}\, dz$$

when
(a) z_0 lies inside Γ, and
(b) z_0 lies outside Γ.

Section 3: Geometry

3.1 What is the locus of a point which moves in the plane such that the product of the squares of its distances from the coordinate axes is a positive constant?

3.2 Let

$$x(t) = \frac{1-t^2}{1+t^2} \text{ and } y(t) = \frac{2t}{1+t^2}.$$

What curve does this represent as t varies over $[-1,1]$?

3.3 Consider the line $2x - 3y + 1 = 0$ and the point $P = (1,2)$. Pick out the points that lie on the same side of this line as P.
(a) $(-1,0)$
(b) $(-2,1)$
(c) $(0,0)$

3.4 Consider the points $A = (0,1)$ and $B = (2,2)$ in the plane. Find the coordinates of the point P on the x-axis such that the segments AP and BP make the same angle with the normal to the x-axis at P.

3.5 Let $K = \{(x,y) \mid |x| + |y| \leq 1\}$. Let $P = (-2,2)$. Find the point in K which is closest to P.

3.6 Let S be the sphere in $\mathbb{R}^3$ with centre at the origin and of radius R. Write down the unit outward normal vector to S at a point (x_1, x_2, x_3) on S.

3.7 Pick out the sets which are bounded:
(a) $\{(x,y) \mid x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1\}$.
(b) $\{(x,y) \mid (x+y)(x-y) = 2\}$.
(c) $\{(x,y) \mid x + 2y \geq 2,\ 2x + 5y \leq 10,\ x \geq 0,\ y \geq 0\}$.

3.8 Find the length of the radius of the circle obtained by the intersection of the sphere

$$x^2 + y^2 + z^2 - 2x - 4y - 6z - 2 = 0$$

and the plane $x + 2y + 2z - 20 = 0$.

3.9 Let λ_1 and λ_2 be the eigenvalues of the matrix

$$\begin{bmatrix} a & h \\ h & b \end{bmatrix}.$$

Assume that $\lambda_1 > \lambda_2 > 0$. Write down the lengths of the semi-axes of the ellipse

$$ax^2 + 2hxy + by^2 \;=\; 1$$

as functions of λ_1 and λ_2.

3.10 Let V be the number of vertices, E, the number of edges and F, the number of faces of a polyhedron in $\mathbb{R}^3$. Write down the values of V, E, F and $V - E + F$ for the following polyhedra:
(a) a tetrahedron.
(b) a pyramid on a square base.
(c) a prism with a triangular cross section.