NATIONAL BOARD FOR HIGHER MATHEMATICS

M. A. and M.Sc. Scholarship Test

September 23, 2006

Time Allowed: 150 Minutes

Maximum Marks: 45

Please read, carefully, the instructions on the following page

INSTRUCTIONS TO CANDIDATES

- Please ensure that this question paper booklet contains 10 numbered (and printed) pages. The reverse of each printed page is blank and can be used for rough work.
- There are three parts to this test: Algebra, Analysis and Geometry. Each part consists of **15** questions adding up to **45** questions in all.
- Answer each question, as directed, in the space provided for it in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so do not answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a),(b) and (c)).
- Points will be awarded in the above questions only if **all** the correct choices are made. There will be no partial credit.
- $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively. The symbol I will denote the identity matrix of appropriate order.

SECTION 1: ALGEBRA

1.1 Compute $(\sqrt{3}+i)^{14}+(\sqrt{3}-i)^{14}$ (Hint: Use De Moivre's theorem).

1.2 Let $p(x)$ be the polynomial $x^3-11x^2+ax-36$, where a is a real number. Assume that it has a positive root which is the product of the other two roots. Find the value of a.

1.3 Identify which of the following groups (if any) is cyclic:
(a) $\mathbb{Z}_8 \oplus \mathbb{Z}_8$
(b) $\mathbb{Z}_8 \oplus \mathbb{Z}_9$
(c) $\mathbb{Z}_8 \oplus \mathbb{Z}_{10}$.

1.4 In each of the following examples determine the number of homomorphisms between the given groups:
(a) from $\mathbb{Z}$ to $\mathbb{Z}_{10}$;
(b) from $\mathbb{Z}_{10}$ to $\mathbb{Z}_{10}$;
(c) from $\mathbb{Z}_8$ to $\mathbb{Z}_{10}$.

1.5 Let S_7 be the group of permutations on 7 symbols. Does S_7 contain an element of order 10? If the answer is "yes", then give an example.

1.6 Let G be a finite group and H be a subgroup of G. Let $O(G)$ and $O(H)$ denote the orders of G and H respectively. Identify which of the following statements are necessarily true.
(a) If $O(G)/O(H)$ is a prime number then H is normal in G.
(b) If $O(G) = 2O(H)$ then H is normal in G.
(c) If there exist normal subgroups A and B of G such that $H = \{ab \mid a \in A, b \in B\}$ then H is normal in G.

1.7 Which of the following statements are true?
(a) There exists a finite field in which the additive group is not cyclic.
(b) If F is a finite field, there exists a polynomial p over F such that $p(x) \neq 0$ for all $x \in F$, where 0 denotes the zero in F.
(c) Every finite field is isomorphic to a subfield of the field of complex numbers.

1.8 Let V be a vector space of dimension 4 over the field $\mathbb{Z}_3$ with 3 elements. What is the number of one-dimensional vector subspaces of V?

1.9 Let V be a vector space of dimension $d < \infty$, over $\mathbb{R}$. Let U be a vector subspace of V. Let S be a subset of V. Identify which of the following statements is true:
(a) If S is a basis of V then $U \cap S$ is a basis of U.
(b) If $U \cap S$ is a basis of U and $\{s + U \in V/U \mid s \in S\}$ is a basis of V/U then S is a basis of V.
(c) If S is a basis of U as well as V then the dimension of U is d.

1.10 Let $M(n, \mathbb{R})$ be the vector space of $n \times n$ matrices with real entries. Let U be the subset of $M(n, \mathbb{R})$ consisting $\{(a_{ij}) \mid a_{11} + a_{22} + \ldots + a_{nn} = 0\}$. Is it true that U is a vector subspace of V over $\mathbb{R}$? If so what is its dimension?

1.11 Let A be a 3×3 matrix with complex entries, whose eigenvalues are $1, i$ and $-2i$. If $A^{-1} = aA^2 + bA + cI$, where I is the identity matrix, with $a, b, c \in \mathbb{C}$, what are the values of a, b and c?

1.12 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation of $\mathbb{R}^n$, where $n \geq 3$, and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be the eigenvalues of T. Which of the following statements are true?
(a) If $\lambda_i = 0$, for some $i = 1, \ldots, n$, then T is not surjective.
(b) If T is injective, then $\lambda_i = 1$ for some i, $1 \leq i \leq n$.
(c) If there is a 3-dimensional subspace U of V such that $T(U) = U$, then $\lambda_i \in \mathbb{R}$ for some i, $1 \leq i \leq n$.

1.13 Let $p(x) = a_0 + a_1 + \cdots + a_n x^n$ be the characteristic polynomial of a $n \times n$ matrix A with entries in $\mathbb{R}$. Then which of the following statements is true?
(a) $p(x)$ has no repeated roots.
(b) $p(x)$ can be expressed as a product of linear polynomials with real coefficients.
(c) If $p(x)$ can be expressed as a product of linear polynomials with real coefficients then there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

1.14 Let $\mathbb{Z}_n$ be the ring of integers modulo n, where n is an integer ≥ 2. Then complete the following:
(a) If $\mathbb{Z}_n$ is a field then n is ... .
(b) If Z_n is an integral domain then n is ... .
(c) If there is an injective ring homomorphism of $\mathbb{Z}_5$ to $\mathbb{Z}_n$ then n is ... .

1.15 Let $C[0,1]$ be the ring of continuous real-valued functions on $[0,1]$, with addition and multiplication defined pointwise. For any subset S of $C[0,1]$ let $Z(S) = \{f \in C[0,1] \mid f(x) = 0 \text{ for all } x \in S\}$. Then which of the following statements are true?
(a) If $Z(S)$ is an ideal in $C[0,1]$ then S is closed in $[0,1]$.
(b) If $Z(S)$ is a maximal ideal then S has only one point.
(c) If S has only one point then $Z(S)$ is a maximal ideal.

SECTION 2: ANALYSIS

2.1 Evaluate:

$$\lim_{\theta \to \frac{\pi}{2}} (1 - 5\cot\theta)^{\tan\theta}.$$

2.2 Evaluate:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \cos\left(\frac{\pi k}{2n}\right).$$

2.3 Let $x > 0$. Define

$$f(x) = \int_0^x \frac{\sin xy}{y}\, dy.$$

Evaluate $f'(x)$ as a function of x.

2.4 What is the relation between the height h and the radius r of a right circular cylinder of fixed volume V and minimal total surface area?

2.5 Find the coefficient of x^7 in the Taylor series expansion of the function $f(x) = \sin^{-1} x$ around 0 in the interval $-1 < x < 1$.

2.6 Find the minimum value of the function:

$$f(x, y) = x^2 + 5y^2 - 6x + 10y + 6.$$

2.7 Find the interval of convergence of the series:

$$(x+1) - \frac{(x+1)^2}{4} + \frac{(x+1)^3}{9} - \frac{(x+1)^4}{16} + \ldots$$

2.8 For what values of p does the following series converge?

$$1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots$$

2.9 Pick out the series which are absolutely convergent:
(a)
$$\sum_{n=1}^{\infty}(-1)^n \frac{\cos n\alpha}{n^2}$$
where $\alpha \in \mathbb{R}$ is a fixed real number.
(b)
$$\sum_{n=1}^{\infty}(-1)^n \frac{n \log n}{e^n}.$$
(c)
$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+2}.$$

2.10 Pick out the functions which are continuous at least at one point in the real line:
(a)
$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$
(b)
$$f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$
(c)
$$f(x) = \begin{cases} \sin \pi x & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

2.11 Pick out the functions which are uniformly continuous:
(a)
$$f(x) = \frac{1}{x}, \ x \in]0,1[.$$
(b)
$$f(x) = \frac{\sin x}{x}, \ x \in]0,1[.$$
(c)
$$f(x) = \sin^2 x, \ x \in \mathbb{R}.$$

2.12 Let $\mathcal{C}^1(\mathbb{R})$ denote the set of all continuously differentiable real valued functions defined on the real line. Define

$$A = \{f \in \mathcal{C}^1(\mathbb{R}) \mid f(0) = 0, f(1) = 1, |f'(x)| \leq 1/2 \text{ for all } x \in \mathbb{R}\}$$

where f' denotes the derivative of the function f. Pick out the true statement:
(a) A is an empty set.
(b) A is a finite and non-empty set.
(c) A is an infinite set.

2.13 Let ω_i, $1 \leq i \leq 7$ denote the seventh roots of unity. Evaluate:

$$\Pi_{i=1}^{7}\omega_i.$$

2.14 Pick out the true statements:
(a) $|\sin z| \leq 1$ for all $z \in \mathbb{C}$.
(b) $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$.
(c) $\sin 3z = 3\sin z - 4\sin^3 z$ for all $z \in \mathbb{C}$.

2.15 Evaluate:

$$\int_{\{|z|=2\}} \frac{dz}{(z-1)^3}.$$

SECTION 3: GEOMETRY

3.1 Write down the equation of the locus of a point which moves in the xy-plane so that it is equidistant from the straight lines $y = x$ and $y = -x$.

3.2 What is the shape of the locus of a point which moves in the plane so that it is equidistant from a given point A and a given straight line ℓ (which does not contain the point A)?

3.3 What is the area of a quadrilateral in the xy-plane whose vertices are $(0, 0), (1, 0), (2, 3)$ and $(0, 1)$?

3.4 What is the surface area of the sphere whose equation is given by

$$x^2 + y^2 + z^2 - 4x + 6y - 2z + 13 \;=\; 0?$$

3.5 What is the number of points of intersection of the curves

$$(x^2 + y^2 + 1)(x^2 + y^2 - 2x - 4y + 1) \;=\; 0 \text{ and } x^2 + y^2 - 2x - 2y - 2 \;=\; 0?$$

3.6 The plane $m_1(x - 1) + m_2(y - 2) + m_3(z - 3) = 0$ is tangent to the surface $x^3 + y^3 - z^3 + 3xyz = 0$ at the point $(1, 2, 3)$. What are the values of $m_i, 1 \leq i \leq 3$ such that $\sum_{i=1}^{3} m_i^2 = 1$?

3.7 Find the area enclosed by the circle formed by the intersection of the sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z = 1$ and the plane $x + y + z = 1$.

3.8 Find the lengths of the semi-axes of the ellipse

$$2x^2 + 2xy + 2y^2 \;=\; 1.$$

3.9 Let A, B, C and D be the vertices (in clockwise order) of a rectangle in the xy-plane. Let $f(x, y) = ax + by$ for some fixed real numbers a and b. Given that $f(A) = 5, f(B) = f(D) = 10$, find $f(C)$ (here, if a point $P = (u, v)$, we write $f(P)$ for $f(u, v)$).

3.10 Let $A_i = (x_i, y_i), 1 \leq i \leq 3$ be the vertices of a triangle in the xy-plane. Then, given any point $P = (x, y)$ inside the triangle, we can find three numbers $\lambda_i = \lambda_i(x, y), 1 \leq i \leq 3$ such that

$$\begin{gathered} 0 \leq \lambda_i \leq 1, \quad \text{for all } 1 \leq i \leq 3, \\ \lambda_1 + \lambda_2 + \lambda_3 = 1, \\ x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \ \text{ and } y = \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3. \end{gathered}$$

If $A_1 = (0, 0), A_2 = (1, 0)$ and $A_3 = (0, 1)$, write down $\lambda_i, 1 \leq i \leq 3$ as functions of x and y.

3.11 Consider the points $A = (0, 2)$ and $B = (1, 1)$ in the xy-plane. Consider all possible paths APB where P is an arbitrary point on the x-axis and AP and PB are straight line segments. Find the coordinates of the point P such that the length of the path APB is shortest amongst all such possible paths.

3.12 What curve does the following equation represent in polar coordinates:

$$\frac{2}{r} = 1 + \frac{1}{2} \cos \theta ?$$

3.13 Find the angle between the planes $2x - y + z = 6$ and $x + y + 2z = 3$.

3.14 Which of the following equations represent bounded sets in the xy-plane?
(a) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.
(b) $xy = 1$
(c) $17x^2 - 12xy + 8y^2 + 46x - 28y + 17 = 0$.

3.15 Let P_n be a regular polygon of n sides inscribed in a circle of radius a, where $a > 0$. Let L_n and A_n be the perimeter and area of P_n respectively. Evaluate:

$$\lim_{n \to \infty} \frac{L_n^2}{A_n}.$$